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NOTE ON THE PRINCIPLE OF STATIONARY COMPLEMENTARY ENERGY APPLIED TO FREE VIBRATION OF AN ELASTIC BODY

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Abstract—A free vibration problem of an elastic body is treated in the present paper from the viewpoint of variational formulations. The principle of stationary potential energy is written for the problem. Then, it is generalized through the use of Lagrange's undetermined multipliers and the principle of stationary complementary energy is shown to be derivable from the generalized principle.

1. INTRODUCTION

IT WAS shown in References [1] and [2] that the principle of stationary complementary energy might be extended to eigenvalue problems such as free vibration and stability of elastic bodies. The principle was introduced and proved by Reissner [3] for a problem in which loads, stresses and displacements are simple harmonic functions of time. However, little mention has been made of a systematic derivation of the principle.

The purpose of the present paper is to show that the principle of stationary complementary energy can be derived in a systematic way from the principle of stationary potential energy through the use of Lagrange multipliers. A formulation is made for a free vibration problem of a three-dimensional elastic body, to begin with. Then, it is applied to free lateral vibrations of an elastic beam. A special mention is made of the Rayleigh–Ritz method applied to the principle of stationary complementary energy.

2. FORMULATION OF THE PROBLEM AND THE PRINCIPLE OF STATIONARY POTENTIAL ENERGY

Let a free vibration problem of an elastic body be defined by allowing the body to be mechanically free on part S_1 of the surface boundary and geometrically fixed on the remaining part S_2. The problem is confined to small displacement theory. Then, all the equations defining the problem are linear and the stresses, strains and displacements of the body behave sinusoidally with respect to time. Consequently, if we denote the amplitudes of stresses, strains and displacements by σ_{ij}, ε_{ij} and u_i, respectively, we have for the equations of motion:

$$\sigma_{ij,j}+\lambda\rho u_i = 0, \qquad i = 1,2,3, \tag{1}$$

where $\sigma_{ij} = \sigma_{ji}$. The summation convention is used throughout the present and next sections. In equations (1), $\lambda = \omega^2$, where ω is the natural circular frequency and ρ is the density of the material of the body. The boundary conditions are given by

$$\sigma_{ij}n_j = 0, \qquad i = 1,2,3 \quad \text{on} \quad S_1, \tag{2}$$

and

$$u_i = 0, \qquad i = 1,2,3 \quad \text{on} \quad S_2, \tag{3}$$

n_j being the direction cosines of the unit normal drawn outwards on the boundary. For the sake of simplicity, we shall confine our subsequent discussion to a free vibration problem in which all the eigenvalues are non-zero and positive.

From equations (1) and (2) we have

$$-\iiint_V (\sigma_{ij,j}+\lambda\rho u_i)\delta u_i \, \mathrm{d}V + \iint_{S_1} \sigma_{ij}n_j\delta u_i \, \mathrm{d}S = 0, \tag{4}$$

where $\mathrm{d}V$ and $\mathrm{d}S$ are elements of the volume and surface area, respectively. After some calculation, equation (4) can be written as follows:

$$\iiint_V \sigma_{ij}\delta\varepsilon_{ij} \, \mathrm{d}V - \lambda\iiint_V \rho u_i\delta u_i \, \mathrm{d}V = 0, \tag{5}$$

where equations (3) have been used in the derivation and

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}). \tag{6}$$

If the relations between amplitudes of stresses and strains are given by

$$\sigma_{ij} = C_{ijmn}\varepsilon_{mn}, \qquad C_{ijmn} = C_{mnij}, \tag{7}$$

or conversely by

$$\varepsilon_{ij} = k_{ijmn}\sigma_{mn}, \qquad k_{ijmn} = k_{mnij}, \tag{8}$$

we are assured of the existence of the potential energy function $A(\varepsilon_{ij})$ and the complementary energy function $B(\sigma_{ij})$ defined by

$$\delta A = \sigma_{ij}\delta\varepsilon_{ij}, \qquad \delta B = \varepsilon_{ij}\delta\sigma_{ij}, \tag{9}$$

as follows:

$$A = \tfrac{1}{2}C_{ijmn}\varepsilon_{ij}\varepsilon_{mn}, \qquad B = \tfrac{1}{2}k_{ijmn}\sigma_{ij}\sigma_{mn}. \tag{10}$$

For later convenience, a notation $A(u_i)$ is introduced here to indicate that the arguments ε_{ij} of A are replaced by u_i through the use of equations (6). With these preliminaries, we obtain the functional for the principle of stationary potential energy as follows:

$$\Pi = \iiint_V A(u_i) \, \mathrm{d}V - \tfrac{1}{2}\lambda\iiint_V \rho u_i u_i \, \mathrm{d}V, \tag{11}$$

where the quantities subject to variation are u_i under the subsidiary conditions (3), where λ is treated as a parameter not subject to variation. It is well known that the principle (11) is equivalent to finding, among admissible functions u_i, those which make the Rayleigh quotient defined by

$$\lambda = \iiint_V A(u_i) \, \mathrm{d}V \Big/ \tfrac{1}{2}\iiint_V \rho u_i u_i \, \mathrm{d}V \tag{12}$$

stationary and that the stationary values of λ provide the eigenvalues [4, 5, 6].

3. DERIVATION OF THE PRINCIPLE OF STATIONARY COMPLEMENTARY ENERGY

The principle (11) can be generalized in a manner similar to the developments for boundary value problems [7, 8, 9], and we obtain one of the generalizations as follows [10]:

$$\prod_{\mathrm{I}} = \iiint_V \{A(\varepsilon_{ij}) - \tfrac{1}{2}\lambda\rho u_i u_i - [\varepsilon_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i})]\sigma_{ij}\}\,\mathrm{d}V - \iint_{S_2} p_i u_i\,\mathrm{d}S, \qquad (13)$$

where σ_{ij} and p_i are Lagrange multipliers and the quantities subject to variation are σ_{ij}; ε_{ij}; u_i; and p_i under no subsidiary conditions. The stationary conditions with respect to these quantities are shown to be equations (6); (7); (1), (2) and

$$p_i = \sigma_{ij} n_j \quad \text{on} \quad S_2; \qquad (14)$$

and (3), respectively.

Several variational principles can be derived from the generalized principle. Here, we shall derive two expressions for the principle of stationary complementary energy under the assumption that $\lambda \neq 0$. First, it is shown that elimination of ε_{ij} by the use of equations (7) and a simple calculation by the use of equations (1), (2) and (14) lead to a transformation of the functional (13) as follows:

$$\prod_{\mathrm{c}} = \iiint_V B(\sigma_{ij})\,\mathrm{d}V - \tfrac{1}{2}\lambda \iiint_V \rho u_i u_i\,\mathrm{d}V, \qquad (15)$$

where the quantities subject to variation are σ_{ij} and u_i under the subsidiary conditions (1) and (2). The stationary conditions are shown to be the conditions of compatibility (6) and the geometrical boundary conditions (3). The functional (15) is equivalent to that introduced by Reissner [3].

Further elimination of the displacement components u_i by the use of equations (1) and a multiplication by a factor $(-\lambda)$ lead to a transformation of the functional (15) as follows:

$$\hat{\prod_{\mathrm{c}}} = \tfrac{1}{2} \iiint_V \rho U_i U_i\,\mathrm{d}V - \lambda \iiint_V B(\sigma_{ij})\,\mathrm{d}V, \qquad (16)$$

where U_i is given by

$$U_i \equiv -(1/\rho)\sigma_{ij,j} \qquad (17)$$

and the quantities subject to variation are the stress components under the subsidiary conditions (2). The stationary conditions are shown to be equivalent to the conditions of compatibility and the geometrical boundary conditions on S_2. It is seen that the principle (16) is equivalent to finding, among admissible σ_{ij}, those which make a Rayleigh quotient defined by

$$\lambda = \tfrac{1}{2} \iiint_V \rho U_i U_i\,\mathrm{d}V \Big/ \iiint_V B(\sigma_{ij})\,\mathrm{d}V \qquad (18)$$

stationary and that the stationary values of λ provide the eigenvalues of the transformed problem.

Mention should be made of the transformed eigenvalue problem derived from the functional (16). It is observed that the governing equations of the transformed problem contain not only all the non-zero eigenvalues of the original problem, but also an additional eigenvalue $\lambda = 0$ and that all the sets of σ_{ij} which satisfy

$$\sigma_{ij,j} = 0 \tag{19}$$

and equations (2) constitute a family of eigenfunctions belonging to $\lambda = 0$. This additional eigenvalue makes the minimum value of the Rayleigh quotient defined by equation (18) zero, as is shown by substituting one of the family of eigenfunctions into the right hand side of equation (18).

4. APPLICATION TO FREE LATERAL VIBRATION OF A BEAM

We shall take as an example a beam clamped at one end $x = 0$, and simply supported at the other end $x = l$. The principle of stationary potential energy for the present problem is given by

$$\prod = \tfrac{1}{2}\int_0^l EI(w'')^2\,\mathrm{d}x - \tfrac{1}{2}\lambda\int_0^l mw^2\,\mathrm{d}x, \tag{20}$$

where EI, w and m are the bending rigidity, deflection and mass per unit span of the beam, respectively, and $(\ )' = \mathrm{d}(\ )/\mathrm{d}x$. In the functional (20), the quantity subject to variation is w under the subsidiary conditions

$$w(0) = w(l) = w'(0) = 0. \tag{21}$$

We denote eigenvalues and corresponding eigenfunctions by

$$\lambda_i, \qquad {}_w\phi_i(x); \qquad i = 1, 2, 3, \ldots \tag{22}$$

in the ascending order of magnitude such that $0 < \lambda_1 < \lambda_2 < \ldots$.

By the introduction of an auxiliary function $\kappa(x)$ defined by

$$\kappa = w'' \tag{23}$$

and Lagrange multipliers $M(x)$, P, Q and R, the functional (20) can be generalized as follows:

$$\prod\nolimits_{\mathrm{I}} = \tfrac{1}{2}\int_0^l EI\kappa^2\,\mathrm{d}x - \tfrac{1}{2}\lambda\int_0^l mw^2\,\mathrm{d}x + \int_0^l (\kappa - w'')M\,\mathrm{d}x + Pw(0) + Qw(l) + Rw'(0), \tag{24}$$

where the quantities subjected to variation are κ, w, M, P, Q and R under no subsidiary conditions. The stationary conditions with respect to κ and w are shown to be

$$EI\kappa + M = 0, \tag{25}$$

$$M'' + \lambda mw = 0, \tag{26}$$

$$P - M'(0) = 0, \qquad Q + M'(l) = 0, \qquad R + M(0) = 0, \tag{27}$$

$$M(l) = 0. \tag{28}$$

Eliminating κ, P, Q and R by the use of equations (25) and (27), and with the aid of equations (26) and (28), we may transform the functional (24) as follows:

$$\prod\nolimits_{c} = \frac{1}{2}\int_0^l \frac{M^2}{EI}\,\mathrm{d}x - \frac{1}{2}\lambda\int_0^l mw^2\,\mathrm{d}x, \tag{29}$$

where the quantities subject to variation are M and w under the subsidiary conditions (26) and (28).

Further elimination of w from the functional (29) by the use of equation (26) and a multiplication by a factor $(-\lambda)$ lead to

$$\hat{\Pi}_c = \frac{1}{2}\int_0^l mW^2\,\mathrm{d}x - \frac{1}{2}\lambda\int_0^l \frac{M^2}{EI}\,\mathrm{d}x, \tag{30}$$

where

$$W \equiv -(1/m)M''. \tag{31}$$

In the functional (30), the quantity subject to variation is M under the subsidiary condition (28). The governing equations of this transformed eigenvalue problem are then obtained from the functional (30) as follows:

$$W'' + (\lambda/EI)M = 0, \tag{32}$$

$$W = 0, \qquad W' = 0 \quad \text{at} \quad x = 0, \tag{33}$$

$$W = 0, \qquad M = 0 \quad \text{at} \quad x = l. \tag{34}$$

where W is written in terms of M by the use of equation (31). It is seen that the eigenvalues and the corresponding eigenfunctions of this transformed problem are given by

$$\lambda_i, \qquad {}_M\phi_i(x); \qquad i = 0, 1, 2, 3, \ldots \tag{35}$$

where

$$\lambda_0 = 0, \qquad {}_M\phi_0(x) = x - l, \tag{36}$$

and

$${}_M\phi_i = -EI_w\phi_i''; \qquad i = 1, 2, 3, \ldots. \tag{37}$$

Thus, the transformed problem contains not only all the eigenvalues and eigenfunctions of the original eigenvalue problem, but also an additional eigenvalue and corresponding eigenfunction given by equations (36).

5. RAYLEIGH-RITZ METHOD APPLIED TO THE PRINCIPLE OF STATIONARY POTENTIAL ENERGY

It is well-established that when variational formulations are available, the Rayleigh–Ritz method provides a powerful tool for obtaining approximate solutions [4, 5, 6]. When the method is applied to the principle (20), we may assume

$$w = \sum_{i=1}^{n} c_i w_i(x), \tag{38}$$

where w_i is a coordinate function which satisfies equation (21). Substituting equation (38) into the functional (20) and requiring that

$$\partial\Pi/\partial c_i = 0; \qquad i = 1, 2, \ldots, n, \tag{39}$$

we obtain a characteristic equation which determines approximate eigenvalues

$$\Lambda_1, \Lambda_2, \ldots, \Lambda_n, \tag{40}$$

where it is specified that $\Lambda_1 \leq \Lambda_2 \leq \ldots \leq \Lambda_n$. They provide upper bounds of the exact eigenvalues as follows [5, 7, 11]:

$$\lambda_i \leq \Lambda_i; \qquad i = 1, 2, \ldots, n. \tag{41}$$

A numerical example is shown by taking

$$w = (c_1 x^2 + c_2 x^3)(x - l) \tag{42}$$

for a beam with constant EI and m. Numerical results are shown in Table 1 and compared with the exact eigenvalues.

TABLE 1. EXACT AND APPROXIMATE EIGENVALUES

$\omega_i = k_i\sqrt{(EI/ml^4)}$

	Exact eigenvalues	Approximate eigenvalues	
		Rayleigh–Ritz method applied to the functional (20)	Modified Rayleigh–Ritz method applied to the functional (29)
k_1	15·42	15·45	15·42
k_2	49·96	75·33	51·93

6. RAYLEIGH–RITZ METHOD APPLIED TO THE PRINCIPLE OF STATIONARY COMPLEMENTARY ENERGY

Next, we shall apply the Rayleigh–Ritz method to the principle of stationary complementary energy. Since the principle has two expressions as given by the functionals (29) and (30), we may have two approaches for obtaining approximate solutions.

The first approach, which is sometimes called the modified Rayleigh–Ritz method [12], employs the functional (29) and its outline is as follows: We shall choose w as given by equation (38) where the coordinate functions w_i are so chosen as to satisfy equation (21). As the derivation of the functional (29) shows, it is not necessary for the coordinate functions to satisfy equation (21). However, this imposition is very desirable for improving the accuracy of approximate solutions and is essential for obtaining inequality relations given later by equations (47). We substitute equation (38) into equation (26) and perform integrations with the boundary condition (28) to obtain

$$(1/\lambda)M = c(x-l) - \sum_{i=1}^{n} c_i \int_x^l \left[\int_\eta^l m(\xi) w_i(\xi) \, d\xi \right] d\eta, \tag{43}$$

where c is an integration constant. Substituting equation (38) and (43) into the functional (29) and requiring that

$$\partial \Pi_c / \partial c = 0, \tag{44}$$

and

$$\partial \Pi_c / \partial c_i = 0, \qquad i = 1, 2, \ldots, n, \tag{45}$$

we obtain a characteristic equation which determines non-zero approximate eigenvalues

$$\Lambda_1^*, \Lambda_2^*, \ldots, \Lambda_n^*, \tag{46}$$

where they are defined such that $\Lambda_1^* \leq \Lambda_2^* \leq \ldots \Lambda_n^*$. A numerical example is shown by taking w as given by equations (42) for a beam with constant EI and m. Numerical results are shown in Table 1.

Mention is made here of the modified Rayleigh–Ritz method. We see that the inclusion of the term $c(x-l)$ in equation (43) and the requirement of equation (44) are equivalent to obtaining the *exact* beam deflection due to the inertial loading $\lambda m \Sigma c_i w_i$. Thus, this method is equivalent to the Grammel's method in which the *exact* deflection due to assumed modes is obtained by the use of the Green's function or the so-called influence function [13, 14, 15, 16]. It is stated in Reference [13] that when the same assumed modes (38) are employed, we have the following inequality relations:

$$\lambda_i \leq \Lambda_i^* \leq \Lambda_i, \qquad i-1, 2, \ldots, n. \tag{47}$$

The above relations were proved by Grammel [13] for the Galerkin method in which the coordinate functions are chosen so as to satisfy all the boundary conditions, namely equation (21) as well as

$$EIw'' = 0 \quad \text{at} \quad x = l. \tag{48}$$

A rigorous proof that the relations (47) hold for the modified Rayleigh–Ritz method in which the coordinate functions w_i are chosen so as to satisfy the geometrical boundary conditions (21), but not necessarily the mechanical boundary conditions (48), has been recently given by Fujita [17].

The modified Rayleigh–Ritz method can be applied to two- or three-dimensional free vibration problems of an elastic body in a similar manner, if the role of the method is interpreted as to obtain *exact* deformations due to assumed loadings by the use of the complementary energy principle. It is obvious that the same technique can be applied to other eigenvalue problems [1, 2].

When the Rayleigh–Ritz method is applied to the functional (30), we must remember that the additional eigenvalue given by equation (36) is contained in this transformed problem. Consequently, we shall take

$$M = a_{0M}\phi_0(x) + \sum_{i=1}^{n} a_i M_i(x), \tag{49}$$

where $M_i(x)$ is a coordinate function which satisfies equation (28). Substituting equation (49) into the functional (30), and requiring that

$$\partial \hat{\Pi}_c / \partial a_i = 0, \qquad i = 0, 1, 2, \ldots, n, \tag{50}$$

we obtain $(n+1)$ approximate eigenvalues

$$0, \Lambda_1^{**}, \Lambda_2^{**}, \ldots, \Lambda_n^{**}, \tag{51}$$

where it is specified that $0 < \Lambda_1^{**} \leq \Lambda_2^{**} \leq \ldots \leq \Lambda_n^{**}$. Then, following a proof similar to that which leads to the inequality relations (41), we find that

$$\lambda_i \leq \Lambda_i^{**}, \qquad i = 1, 2, \ldots, n. \tag{52}$$

A numerical example is shown by taking

$$M = a_0(x-l)+(a_1x+a_2x^2)(x-l), \tag{53}$$

and requiring that

$$\partial\hat{\Pi}_c/\partial a_i = 0, \qquad i = 0, 1, 2. \tag{54}$$

After some calculation, we obtain approximate eigenvalues as follows:

$$0, \qquad (17{\cdot}54)^2EI/ml^4, \qquad (70{\cdot}09)^2EI/ml^4, \tag{55}$$

of which the non-zero values provide upper bounds of the eigenvalues λ_1 and λ_2.

The employment of the term $a_{0M}\phi_0(x)$ in equation (49) is essential for obtaining the bounds formulae (52), because if this term would be discarded, we would find that the lowest approximate eigenvalue obtained by the Rayleigh–Ritz method has a non-zero positive value which is possibly smaller than λ_1, thus providing an upper bound of the additional eigenvalue λ_0. For example, by taking

$$M = (a_1x+a_2x^2)(x-l), \tag{56}$$

and requiring that

$$\partial\hat{\Pi}_c/\partial a_i = 0, \qquad i = 1, 2, \tag{57}$$

we obtain approximate eigenvalues as follows:

$$(10{\cdot}95)^2EI/ml^4, \qquad (50{\cdot}20)^2EI/ml^4 \tag{58}$$

which provide upper bounds of the eigenvalue λ_0 and λ_1.

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Résumé—Dans cette étude, le problème de la vibration libre d'un corps élastique est traité du point de vue des formulations sujettes à des variations. Le principe de l'énergie potentielle stationnaire est indiqué pour le problème. Il est ensuite généralisé par l'emploi des multiplicateurs indéterminés de Lagrange, et le principe de l'énergie stationnaire complémentaire est indiqué en tant que dérivant du principe généralisé.

Zussammenfassung—Ein freischwing-Problem eines elastischen Körpers wird in vorliegender Abhandlung vom Standpunkt der Variationsformulierung betrachtet. Das Prinzip der stationären Potenzialenergie ist für das Problem geschrieben und durch Verwendung von Lagrange's unbestimmter Multiplikatoren verallgemeinert. Das Prinzip der stationären Komplementärenergie kann vom verallgemeinerten Prinzip abgeleitet werden.

Абстракт—Проблема свободного колебания эластичного тела расследованна методом вариационной формулировки. В проблему вписывается принцип стационарной потенциальной енергии, который затем обобщается при помощи неопределенных коэффициентов Лагранжа, причем выявляется, что принцип стационарной дополнитнльой енергии можно извлечь из обобщенного принципа.